Complex analysis for EE, 2012-13, problem set 1

1. Solve:
   (a) $z^2 = i$.
   (b) $z^5 = -12$.
   (c) $z^2 - z + 1 = -i$.
   (d) $z^2 + (\alpha + i\beta)z + (\gamma + i\delta) = 0$, where $\alpha, \beta, \gamma, \delta$ are arbitrary real numbers.

2. The square root function $f(x) = \sqrt{x}$ is defined for $x \in [0, \infty)$ by choosing the positive square root. It satisfies the following two conditions:
   - $f(a)^2 = a$.
   - $f(ab) = f(a)f(b)$.

   Show that there is no way to extend the square root function to the entire complex plane in a way that the previous two conditions hold. That is, there is no consistent way to choose one of the two roots for each number z and call it "$\sqrt{z}$" such that we still have the formula $\sqrt{ab} = \sqrt{a}\sqrt{b}$.

3. Let $\Omega = \{x + iy \mid x, y > 0\}$ be the first quadrant. Let $f(z) = z^2$.
   (a) What is $f(\Omega)$?
   (b) Consider the triangle whose vertices are at the points $1 + i, 1 + 2i, 2 + i$. What is the image of this triangle under f?
   (c) Given two smooth curves which intersect at a point a, the *angle* between them at that point is defined to be the angle between the tangent lines to those curves at this point. Compute the angles between the images under f of the sides of the triangles. How are they related to the angles of the original triangle?

4. (a) Show that $\bar{z} = \dfrac{1}{z}$ if and only if $|z| = 1$.
   (b) Show that if $z \neq 0$ then $\dfrac{z}{\bar{z}}$ is on the unit circle, and any complex number on the unit circle can be written in this form.

5. (a) Show that $\left|\dfrac{a - b}{1 - \bar{a}b}\right| < 1$ for $|a| < 1$ and $|b| < 1$.
   (b) Show that $\left|\dfrac{a - b}{1 - \bar{a}b}\right| = 1$ if either $|a| = 1$ or $|b| = 1$ but not both; what exception must be made for the case $|a| = |b| = 1$?

6. Let $|a_i| < 1$, $\lambda_i \geq 0$, $i = 1, \ldots, n$, and $\lambda_1 + \cdots + \lambda_n = 1$. Show that
$$|\lambda_1 a_1 + \cdots + \lambda_n a_n| < 1.$$

7. (a) Find the equation of the line through c that is perpendicular to the line $z = a + bt$.
   (b) Find the symmetric point of c with respect to the line $z = a + bt$.

8. (a) Find the solution set of the equation $az + b\bar{z} + c = 0$ (the solution set can be a line, a point, or an empty set depending on the values of a, b, and c).

(b) Show that if a circle passes through two distinct points c and d then its center lies on the line that is perpendicular to the line segment from c to d through its center.

9. Let $\omega = \cos 2\pi/n + i\,\sin 2\pi/n$.

   (a) Show that
   $$1 + \omega^h + \omega^{2h} + \cdots + \omega^{(n-1)h} = 0$$
   for any integer h that is not a multiple of n.

   (b) Compute
   $$1 - \omega^h + \omega^{2h} - \cdots + (-1)^{n-1}\omega^{(n-1)h} = 0.$$